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A comment on Gans' stability criterion for steady inviscid helical gas flows

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Gans' criterion is shown to be valid for all inviscid non-diffusive flows where the radial velocity is everywhere zero and the physical variables are functions of the radial distance only. That is, a modified form of the Miles–Howard theorem holds for a large class of helical gas flows.

1. Introduction

Gans (1975) showed that under certain restrictions (e.g. if the ratio of the axial to the azimuthal wavenumber of disturbances is small) the flow of a gas through a rotating pipe is linearly stable if and only if a Richardson number criterion is satisfied. It is shown here that these restrictions may be relaxed and that this criterion, or rather a simple modification of it, is valid for all non-diffusive flows where the velocity is of the form

$$v_0(r)\,\hat{\boldsymbol{\phi}} + w_0(r)\,\hat{\mathbf{z}} \tag{1.1}$$

and where the temperature is also a function of the radius only. The (r, ϕ, z) co-ordinates are cylindrical ones, the z axis coinciding with the axis of the two coaxial circular cylinders (radii a and b, $0 < a < b$) between which the flow occurs. In order to make some ready comparisons, an artificial gravity $g(r)\,\hat{\mathbf{r}}$ is also introduced, whereby the fluid is supposedly repulsed from the axis with a force g per unit mass. Later g will be set equal to zero.

2. The basic equations

The equations of momentum, continuity and compressibility, together with the equation of state of the gas, form the fundamental equations. The system is perturbed infinitesimally, and effects of thermal diffusion and viscosity are ignored, so that small changes in the density occur adiabatically. If the suffix zero denotes the basic state, then the equations governing the perturbations are (where the symbols have their usual meaning)

$$p' + i\Lambda\rho_0 u - (g + v_0^2/r)\,\rho - 2v_0\rho_0 v/r = 0 \quad \text{(radial momentum)},$$

$$imp/r + i\Lambda\rho_0 v + \rho_0(v_0' + v_0/r)\,u = 0 \quad \text{(meridional momentum)},$$

$$ikp + i\Lambda\rho_0 w + \rho_0 w_0' u = 0 \quad \text{(axial momentum)},$$

$$c_0^2(i\Lambda\rho + \rho_0' u) = i\Lambda p + p_0' u \quad \text{(adiabatic compressibility)}$$

and

$$i\Lambda\rho + \rho_0' u + \rho_0(u' + u/r + imv/r + ikw) = 0 \quad \text{(continuity)}.$$

(Cf. Gans (1975, equations 2.6). Extra terms arise here because the angular velocity is a function of the radial distance.) Here c_0 denotes the local sound speed, ρ_0 the basic density and p_0 the basic pressure, which satisfies the equation

$$p_0' = \rho_0(v_0^2/r+g).$$

A prime denotes differentiation with respect to r. The perturbed pressure is assumed to be of the form

$$p(r)\exp(im\phi+ikz-i\lambda t), \quad \text{etc.},$$

where m is an integer and k is real, while λ may take complex values. Also the Doppler frequency Λ is given by

$$\Lambda = \Lambda(r) = -\lambda+mv_0/r+kw_0.$$

Setting $p_1 = -ip$, immediately dropping the suffix and eliminating v, w and ρ leads to the pair of equations

$$u'+\alpha u = Ap, \quad p'+\beta p = Bu, \tag{2.1a,b}$$

where $u(a) = u(b) = 0$, the boundary condition of zero normal velocity at the surfaces of the cylinders. Here

$$\left.\begin{aligned} \alpha &= r^{-1}-2m\tilde{\Omega}/r\Lambda-kw_0'/\Lambda+p_0'/\rho_0c_0^2, \\ A &= \Lambda/c_0^2\rho_0-m^2/r^2\rho_0\Lambda-k^2/\rho_0\Lambda, \\ \beta &= 2m\Omega/r\Lambda-(\Omega^2r+g)/c_0^2, \\ B &= 4\Omega\tilde{\Omega}\rho_0/\Lambda-\Lambda\rho_0+\rho_0(\Omega^2r+g)(\rho_0'/\rho_0-p_0'/p_0c_0^2)/\Lambda. \end{aligned}\right\} \tag{2.2}$$

Also the notion of angular velocity has been introduced, so that

$$\Omega = v_0/r, \quad \tilde{\Omega} = \Omega+\tfrac{1}{2}r\Omega'.$$

3. Remark on the pair of equations (2.1)

If the substitutions $u = FU$ and $p = GP$ are made, where $F(r)$ and $G(r)$ do not vanish in (a, b), then subsequent multiplication of (2.1a) by $\bar{P}/F$ (the bar denotes complex conjugation) and the conjugate of (2.1b) by $U/\bar{G}$ and integration over (a,b) gives

$$\int_a^b dr[|P|^2AG/F+|U|^2\overline{BF/G}-(\alpha+\bar{\beta}+F'/F+\bar{G}'/\bar{G})\bar{P}U] = 0,$$

since $U\bar{P}$ vanishes at the end points. Then since $U\bar{P} = |U|\,|P|\,e^{i\theta}$, say, where θ is real, it follows that the integral here has positive-definite (negative-definite) imaginary part if $\operatorname{Im}(AG/F) \gtrless 0$, and

$$4\operatorname{Im}(AG/F)\operatorname{Im}(\overline{BF/G}) > |\alpha+\bar{\beta}+F'/F+\bar{G}'/\bar{G}|^2 \tag{3.1}$$

everywhere in (a, b). Hence if (3.1) is satisfied it follows that the only solution to (2.1) is the null solution $u = p = 0$. Here F and G may be chosen at will (provided that they are non-vanishing and differentiable). Although they may be chosen such that the right-hand side of (3.1) vanishes, it is simpler for the purpose to set

$F'/F = G'G = -\frac{1}{2}\operatorname{Re}(\alpha+\bar{\beta})$. The conditions then read (with $F = G$) $\operatorname{Im} A \gtrless 0$, and

$$4 \operatorname{Im} A \operatorname{Im} \bar{B} > (\operatorname{Im}(\alpha+\bar{\beta}))^2. \tag{3.2}$$

F need not be found explicitly since it does not appear here.

4. Results

The condition (3.2) derived in the previous section is now applied to (2.1). It is readily seen that the transformation F required can be selected such that it is non-vanishing in (a, b) provided that $\operatorname{Im} \lambda \gtrless 0$. Application of the criterion (3.2) shows that the solution of (2.1) is null if $\operatorname{Im} \lambda \neq 0$, and

$$4(|\Lambda|^2/c_0^2 + m^2/r^2 + k^2)\,[4\Omega\tilde{\Omega} + (\Omega^2 r + g)\,(\rho_0'/\rho_0 - p_0'/c_0^2\rho_0) + |\Lambda|^2] > [kw_0' + 2m(\Omega+\tilde{\Omega})/r]^2 \tag{4.1}$$

everywhere in (a, b). From this result maximum growth rates and a stability criterion follow. If $|\Lambda|$ is replaced in (4.1) by λ_i and the inequality sign is changed to one of equality, then the maximum positive root of the resulting equation is an upper bound to the growth rate. For the stability criterion, it can be seen that it is sufficient for stability that the inequality

$$4\Omega\tilde{\Omega} + (\Omega^2 r + g)\,(\rho_0'/\rho_0 - p_0'/c_0^2\rho_0) > \frac{[kw_0' + 2m(\Omega+\tilde{\Omega})/r]^2}{4(m^2/r^2 + k^2)}$$

holds everywhere in (a, b). The maximum value of the right-hand side (for fixed r) when both m and k range over $(-\infty, \infty)$ occurs when

$$kr/m = w_0'/2(\Omega+\tilde{\Omega}),$$

when it takes the value

$$\tfrac{1}{4}w_0'^2 + (\Omega+\tilde{\Omega})^2.$$

Recalling that

$$\tilde{\Omega} - \Omega = \tfrac{1}{2}r\Omega' = \tfrac{1}{2}(v_0' - v_0/r)$$

it follows that if

$$(g + v_0^2/r)\,(\rho_0'/\rho_0 - p_0'/c_0^2\rho_0) > \tfrac{1}{4}[w_0'^2 + (v_0' - v_0/r)^2] \tag{4.2}$$

everywhere in (a, b) then the flow is stable. In the limit $r \to \infty$, this reduces to the Miles–Howard theorem

$$N^2 > \tfrac{1}{4}(v_0'^2 + w_0'^2). \tag{4.3}$$

N is the Brunt–Väisälä frequency. (The direction of the r axis should be reversed if it is to be in the 'upwards' direction, i.e. against gravity.) Setting $g = 0$, it is seen that in the case v_0/r = constant = Ω inequality (4.2) reduces to

$$|w_0'| < 2(\gamma-1)^{\frac{1}{2}}\,\Omega^2 r/c_0, \tag{4.4}$$

if the temperature is constant, which is Gans' criterion for axial flows. Inequality (4.2) is now seen to be a sufficient condition for stability provided that the radial velocity in the basic flow is zero everywhere, that diffusion and viscous effects are negligible, and that the velocity and temperature depend only on the radial distance from the axis, the radial profiles being differentiable functions of the radius r.

REFERENCE

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